Voxel-Based Representation Learning for Place Recognition Based on 3D Point Clouds
Supplementary Material

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I. Proof of Theorem 1

In this section, we prove that Algorithm 1 (in the main paper) decreases the value of the objective function with each iteration and converges to the global optimal value. But first, we present a lemma:

Lemma 1: Given any two vectors $\mathbf{u}$ and $\tilde{\mathbf{u}}$, the following inequality relation holds: $\|\tilde{\mathbf{u}}\|_2 - \frac{\|\tilde{\mathbf{u}}\|_2^2}{2\|\mathbf{u}\|_2} \leq \|\mathbf{u}\|_2 - \frac{\|\mathbf{u}\|_2^2}{2\|\mathbf{u}\|_2}$.

Proof: We have:

$$(\|\widetilde{\mathbf{u}}\|_2 - \|\mathbf{u}\|_2)^2 \leq 0 \tag{1}$$

$$-\|\widetilde{\mathbf{u}}\|_2^2 - \|\mathbf{u}\|_2^2 + 2\|\widetilde{\mathbf{u}}\|_2\|\mathbf{u}\|_2 \leq 0 \tag{2}$$

$$2\|\widetilde{\mathbf{u}}\|_2\|\mathbf{u}\|_2 - \|\widetilde{\mathbf{u}}\|_2^2 \leq \|\mathbf{u}\|_2^2 \tag{3}$$

$$\|\widetilde{\mathbf{u}}\|_2 - \frac{\|\widetilde{\mathbf{u}}\|_2^2}{2\|\mathbf{u}\|_2} \leq \|\mathbf{u}\|_2 - \frac{\|\mathbf{u}\|_2^2}{2\|\mathbf{u}\|_2} \tag{4}$$

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From Lemma 1, we can derive the following corollary:

Corollary 1: For any two given matrices $\mathbf{U}$ and $\tilde{\mathbf{U}}$, the following inequality relation holds: $\|\tilde{\mathbf{U}}\|_F - \frac{\|\tilde{\mathbf{U}}\|_F^2}{2\|\mathbf{U}\|_F} \leq \|\mathbf{U}\|_F - \frac{\|\mathbf{U}\|_F^2}{2\|\mathbf{U}\|_F}$.

Theorem 1: Algorithm 1 converges to the optimal solution to the optimization problem in Eq. (4) of the main paper

Proof: From Algorithm 1,

$$\begin{aligned}\mathbf{W}(t+1) &= \min_{\mathbf{W}} \|\mathbf{X}^\top\mathbf{W} - \mathbf{Y}\|_F^2 \\ &+\lambda_1 Tr\mathbf{W}^\top\mathbf{D}(t+1)\mathbf{W} + \lambda_2 Tr\mathbf{W}^\top\tilde{\mathbf{D}}(t+1)\mathbf{W}.\end{aligned} \tag{5}$$

Then, we can derive that

$$\begin{aligned}&\mathcal{J}(t+1) + \lambda_1 Tr\mathbf{W}^\top(t+1)\mathbf{D}(t+1)\mathbf{W}(t+1) \\ &+\lambda_2 Tr\mathbf{W}^\top(t+1)\tilde{\mathbf{D}}(t+1)\mathbf{W}(t+1) \\ \leq\ &\mathcal{J}(t) + \lambda_1 Tr\mathbf{W}^\top(t)\mathbf{D}(t+1)\mathbf{W}(t) \\ &+\lambda_2 Tr\mathbf{W}^\top(t)\tilde{\mathbf{D}}(t+1)\mathbf{W}(t),\end{aligned}$$

where $\mathcal{J}(t) = \|\mathbf{X}^\top\mathbf{W}(t) - \mathbf{Y}\|_F^2$.
After substituting the definition of $\mathbf{D}$ and $\tilde{\mathbf{D}}$, we obtain

$$\mathcal{J}(t+1) + \lambda_1 \sum_{i=1}^{v} \frac{\|\mathbf{W}^i(t+1)\|_F^2}{2\|\mathbf{W}^i(t)\|_F} \tag{6}$$

$$+\lambda_2\Big(\sum_{i=1}^{m} \frac{\|\mathbf{W}^i(t+1)\|_F^2}{2\|\mathbf{W}^i(t)\|_F} + \sum_{i=1}^{d} \frac{\|\mathbf{w}^i(t+1)\|_2^2}{2\|\mathbf{w}^i(t)\|_2}\Big)$$

$$\leq \mathcal{J}(t) + \lambda_1 \sum_{i=1}^{v} \frac{\|\mathbf{W}^i(t)\|_F^2}{2\|\mathbf{W}^i(t)\|_F} \tag{7}$$

$$+\lambda_2\Big(\sum_{i=1}^{m} \frac{\|\mathbf{W}^i(t)\|_F^2}{2\|\mathbf{W}^i(t)\|_F} + \sum_{i=1}^{d} \frac{\|\mathbf{w}^i(t)\|_2^2}{2\|\mathbf{w}^i(t)\|_2}\Big)$$

From Lemma 1 and Corollary 1, we can derive that

$$\begin{aligned}&\sum_{i=1}^{v}\|\mathbf{W}^i(t+1)\|_F - \sum_{i=1}^{v}\frac{\|\mathbf{W}^i(t+1)\|_F^2}{2\|\mathbf{W}^i(t)\|_F} \leq \\ &\sum_{i=1}^{v}\|\mathbf{W}^i(t)\|_F - \sum_{i=1}^{v}\frac{\|\mathbf{W}^i(t)\|_F^2}{2\|\mathbf{W}^i(t)\|_F}.\end{aligned} \tag{8}$$

,

$$\begin{aligned}&\sum_{i=1}^{m}\|\mathbf{W}^i(t+1)\|_F - \sum_{i=1}^{m}\frac{\|\mathbf{W}^i(t+1)\|_F^2}{2\|\mathbf{W}^i(t)\|_F} \leq \\ &\sum_{i=1}^{m}\|\mathbf{W}^i(t)\|_F - \sum_{i=1}^{m}\frac{\|\mathbf{W}^i(t)\|_F^2}{2\|\mathbf{W}^i(t)\|_F},\end{aligned} \tag{9}$$

and,

$$\begin{aligned}&\sum_{i=1}^{d}\|\mathbf{w}^i(t+1)\|_2 - \sum_{i=1}^{d}\frac{\|\mathbf{w}^i(t+1)\|_2^2}{2\|\mathbf{w}^i(t)\|_2} \leq \\ &\sum_{i=1}^{d}\|\mathbf{w}^i(t)\|_2 - \sum_{i=1}^{d}\frac{\|\mathbf{w}^i(t)\|_2^2}{2\|\mathbf{w}^i(t)\|_2},\end{aligned} \tag{10}$$

Adding Eq. (6)-(10) on both sides, we have

$$\mathcal{J}(t+1) + \lambda_1 \sum_{i=1}^{v}\|\mathbf{W}^i(t+1)\|_F \tag{11}$$

$$+\lambda_2\Big(\sum_{i=1}^{m}\|\mathbf{W}^i(t+1)\|_F + \sum_{i=1}^{d}\|\mathbf{w}^i(t+1)\|_2\Big)$$

$$\leq \mathcal{J}(t) + \lambda_1 \sum_{i=1}^{v}\|\mathbf{W}^i(t)\|_F \tag{12}$$

$$+\lambda_2\Big(\sum_{i=1}^{m}\|\mathbf{W}^i(t)\|_F + \sum_{i=1}^{d}\|\mathbf{w}^i(t)\|_2\Big)$$

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Eq. (11) decreases the value of the objective function with each iteration. As our objective function is convex, Algorithm 1 converges to the optimal value. Therefore, Algorithm 1 converges to the optimal solution to the optimization problem in Eq. (4) of the main paper. ■